

# A DECOMPOSITION OF FUNCTIONS WITH ZERO MEANS ON CIRCLES

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ABSTRACT It is well known that every Hölder continuous function on the unit circle is the sum of two functions such that one of these functions extends holomorphically into the unit disc and the other extends holomorphically into the complement of the unit disc. We prove that an analogue of this holds for Hölder continuous functions on an annulus  $A$  which have zero averages on all circles contained in  $A$  which surround the hole.

## 1. Introduction and the main results

Given  $a \in \mathbb{C}$ ,  $\rho > 0$  write  $\Delta(a, \rho) = \{\zeta \in \mathbb{C}: |\zeta - a| < \rho\}$  and  $\Delta = \Delta(0, 1)$ . Denote by  $\mathbb{B}$  the open unit ball in  $\mathbb{C}^2$ . A function  $f$  on a set  $K \subset \mathbb{C}$  is called Hölder continuous on  $K$  (with exponent  $\alpha$ ) if there are constants  $M < \infty$  and  $\alpha$ ,  $0 < \alpha < 1$ , such that  $|f(z) - f(w)| \leq M|z - w|^\alpha$  ( $z, w \in K$ ).

Let  $a \in \mathbb{C}$ ,  $\rho > 0$  and let  $f$  be a Hölder continuous function  $f$  on  $b\Delta(a, \rho)$ . It is well known that

$$f = f^+ + f^- \quad (1.1)$$

where  $f^+$  and  $f^-$  are Hölder continuous functions on  $b\Delta(a, \rho)$  such that  $f^+$

$$\text{has a continuous extension to } \overline{\Delta}(a, \rho) \text{ which is holomorphic on } \Delta(a, \rho) \quad (1.2)$$

and  $f^-$

$$\begin{aligned} &\text{has a continuous extension to } [\mathbb{C} \cup \{\infty\}] \setminus \Delta(a, \rho) \text{ which is holomorphic} \\ &\text{on } [\mathbb{C} \cup \{\infty\}] \setminus \overline{\Delta}(a, \rho) \text{ and vanishes at } \infty \end{aligned} \quad (1.3)$$

and this decomposition is unique. In fact,

$$\frac{1}{2\pi i} \int_{b\Delta(a, \rho)} \frac{f(\zeta) d\zeta}{\zeta - z} = \begin{cases} f^+(z) & (z \in \Delta(a, \rho)) \\ -f^-(z) & (z \in [\mathbb{C} \cup \{\infty\}] \setminus \overline{\Delta}(a, \rho)). \end{cases} \quad (1.4)$$

In the present paper we consider functions on the annulus

$$A = \{\zeta \in \mathbb{C}: r_1 \leq |\zeta| \leq r_2\}$$

where  $0 < r_1 < r_2 < \infty$  and ask for a decomposition similar to (1.1). Suppose that  $f$  is a Hölder continuous function on the annulus  $A$ . For every circle  $b\Delta(a, \rho) \subset A$  surrounding the origin we have  $f|_{b\Delta(a, \rho)} = f_{a, \rho}^+ + f_{a, \rho}^-$  where  $f_{a, \rho}^+$  satisfies (1.2) and  $f_{a, \rho}^-$  satisfies (1.3). In general there are no functions  $f^+$  and  $f^-$  on  $A$  such that  $f^+|_{b\Delta(a, \rho)} = f_{a, \rho}^+$  and  $f^-|_{b\Delta(a, \rho)} = f_{a, \rho}^-$  whenever  $b\Delta(a, \rho) \subset A$  surrounds the origin. In the present paper we prove that there are such functions  $f^+$  and  $f^-$  on  $A$  whenever  $f$  satisfies

$$\frac{1}{2\pi} \int_0^{2\pi} f(a + \rho e^{i\theta}) d\theta = 0 \quad (1.5)$$

for every  $b\Delta(a, \rho) \subset A$  which surrounds the origin:

**Theorem 1.1** *Let  $f$  be a Hölder continuous function on  $A$  which satisfies (1.5) whenever  $b\Delta(a, \rho) \subset A$  surrounds the origin. Then  $f = f^+ + f^-$  where  $f^+$  and  $f^-$  are Hölder continuous functions on  $A$  such that for each  $b\Delta(a, \rho) \subset A$  which surrounds the origin,  $f^+|_{b\Delta(a, \rho)}$  satisfies (1.2) and  $f^-|_{b\Delta(a, \rho)}$  satisfies (1.3).*

We will also show that, as in the case of the circle, we can view  $f^+$  and  $f^-$  as the boundary values of functions, holomorphic on appropriate domains. To describe this, we first rewrite the circle case in a form suitable for generalization.

Let  $f$  be a continuous function on  $b\Delta(a, \rho)$ . The idea is to define a new function  $F$  on  $\{(\zeta, \bar{\zeta}): \zeta \in b\Delta(a, \rho)\}$ , that is, on  $b\Delta(a, \rho)$  "lifted" to

$$\Sigma = \{(\zeta, \bar{\zeta}): \zeta \in \mathbb{C}\},$$

by

$$F(\zeta, \bar{\zeta}) = f(\zeta) \quad (\zeta \in b\Delta(a, \rho))$$

[G2] and then to write  $F$  as the sum of boundary values of holomorphic functions.

Let

$$\Lambda_{a, \rho} = \{(z, w) \in \mathbb{C}^2: (z - a)(w - \bar{a}) = \rho^2\}.$$

The intersection  $\Lambda_{a, \rho} \cap \Sigma$  is the circle  $\{(\zeta, \bar{\zeta}): \zeta \in b\Delta(a, \rho)\}$  whose complement in  $\Lambda_{a, \rho}$  has two components,  $\Lambda_{a, \rho}^+$  and  $\Lambda_{a, \rho}^-$  where

$$\begin{aligned} \Lambda_{a, \rho}^+ &= \{(z, w): (z - a)(w - \bar{a}) = \rho^2, 0 < |z - a| < \rho\} \\ \Lambda_{a, \rho}^- &= \{(z, w): (z - a)(w - \bar{a}) = \rho^2, \rho < |z - a|\} \\ &= \{(z, w): (z - a)(w - \bar{a}) = \rho^2, 0 < |w - \bar{a}| < \rho\} \\ &= \{(z, w): (\bar{w}, \bar{z}) \in \Lambda_{a, \rho}^+\}. \end{aligned}$$

The sets  $\Lambda_{a, \rho}^+$  and  $\Lambda_{a, \rho}^-$  are closed one dimensional complex submanifolds of  $\mathbb{C}^2 \setminus \Sigma$  attached to  $\Sigma$  along

$$b\Lambda_{a, \rho}^+ = b\Lambda_{a, \rho}^- = \{(\zeta, \bar{\zeta}): \zeta \in b\Delta(a, \rho)\}.$$

It is easy to see that a continuous function  $h$  on  $b\Delta(a, \rho)$  satisfies (1.2) if and only if the function  $H$  defined on  $b\Lambda_{a, \rho}^+ = b\Lambda_{a, \rho}^-$  by  $H(\zeta, \bar{\zeta}) = h(\zeta)$  ( $\zeta \in b\Delta(a, \rho)$ ) has a bounded continuous extension from  $b\Lambda_{a, \rho}^+$  to  $\Lambda_{a, \rho}^+ \cup b\Lambda_{a, \rho}^+$  which is holomorphic on  $\Lambda_{a, \rho}^+$  [G2]. Similarly,  $h$  satisfies (1.3) if and only if  $H$  has a bounded continuous extension from  $b\Lambda_{a, \rho}^-$  to  $\Lambda_{a, \rho}^- \cup b\Lambda_{a, \rho}^-$  which is holomorphic on  $\Lambda_{a, \rho}^-$  and vanishes at  $\infty$ .

We now pass to functions on  $A$  which we will view as functions on

$$\tilde{A} = \{(\zeta, \bar{\zeta}): \zeta \in A\}.$$

The set  $\tilde{A} \subset \Sigma$  will be a common part of the boundaries of two domains  $\Omega^+(A)$  and  $\Omega^-(A)$  which we now describe. Let  $\Omega^+(A)$  be the union of all  $\Lambda_{a, \rho}^+$  such that  $b\Delta(a, \rho) \subset \text{Int}A$  surrounds the origin. Similarly, let  $\Omega^-(A)$  be the union of all  $\Lambda_{a, \rho}^-$  such that  $b\Delta(a, \rho) \subset \text{Int}A$  surrounds the origin. Clearly,  $\Omega^-(A)$  is the image of  $\Omega^+(A)$  under the reflection

$(z, w) \mapsto (\bar{w}, \bar{z})$ . It turns out that  $\Omega^+(A)$  and  $\Omega^-(A)$  are disjoint domains in  $\mathbb{C}^2 \setminus \Sigma$  attached to  $\Sigma$  along  $\tilde{A}$ . For each  $\zeta \in \text{Int}A$  there are a neighbourhood  $U \subset \Sigma$  of  $(\zeta, \tilde{\zeta})$  and a wedge with the edge  $U$  which is contained in  $\Omega^+(A)$ . An analogous statement holds for  $\Omega^-(A)$ .

**Theorem 1.2** *Let  $f$  be a Hölder continuous function on  $A$  which satisfies (1.5) whenever  $b\Delta(a, \rho) \subset A$  surrounds the origin. There are a bounded continuous function  $G^+$  on  $\Omega^+(A) \cup b\Omega^+(A)$  which is holomorphic on  $\Omega^+(A)$  and a bounded continuous function  $G^-$  on  $\Omega^-(A) \cup b\Omega^-(A)$  which is holomorphic on  $\Omega^-(A)$  such that*

$$f(z) = (1/\bar{z})G^+(z, \bar{z}) + (1/z)G^-(z, \bar{z}) \quad (z \in A).$$

Thus, on  $\tilde{A}$  the function  $F(z, \bar{z}) = f(z)$  is the sum of the boundary values of holomorphic functions  $(1/w)G^+(z, w)$  and  $(1/z)G^-(z, w)$ .

## 2. Fourier coefficients of functions with zero means

Suppose that  $f$  is a continuous function on  $A$ . For each  $r$ ,  $r_1 \leq r \leq r_2$ , let

$$c_k(r) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-ik\theta} f(re^{i\theta}) d\theta \quad (k \in \mathbb{Z})$$

so that  $\sum_{k=-\infty}^{\infty} c_k(r) e^{ik\theta}$  is the Fourier series of the function  $e^{i\theta} \mapsto f(re^{i\theta})$ .

We shall need the following description of the Fourier coefficients of functions with zero means on circles.

**Theorem 2.1** [G1, EK, V] *A continuous function  $f$  on  $A$  satisfies (1.5) for each  $b\Delta(a, \rho) \subset A$  surrounding the origin if and only if*

$$(a) \quad c_0(r) = 0 \quad (r_1 \leq r \leq r_2)$$

$$(b) \quad \text{for each } n \in \mathbb{Z}, \quad n \neq 0, \quad \text{there are numbers } a_{n,0}, a_{n,1}, \dots, a_{n,|n|-1} \text{ such that}$$

$$c_n(r) = r^{-|n|} (a_{n,0} + a_{n,1}r^2 + \dots + a_{n,|n|-1}r^{2(|n|-1)}) \quad (r_1 \leq r \leq r_2).$$

In the rest of this section we assume that  $f$  is a continuous function on  $A$  which satisfies (1.5) for each  $b\Delta(a, \rho) \subset A$  surrounding the origin.

If  $n \geq 1$  then writing  $z = re^{i\theta}$  we get

$$\begin{aligned} c_n(r) e^{in\theta} &= r^{-n} c_n(r) z^n \\ &= (a_{n,0} r^{-2n} + a_{n,1} r^{-2(n-1)} + \dots + a_{n,n-1} r^{-2}) z^n \\ &= \left( \frac{a_{n,0}}{z^n \bar{z}^n} + \frac{a_{n,1}}{z^{n-1} \bar{z}^{n-1}} + \dots + \frac{a_{n,n-1}}{z \bar{z}} \right) z^n \\ &= \frac{1}{\bar{z}} \left( a_{n,0} \frac{1}{\bar{z}^{n-1}} + a_{n,1} \frac{1}{\bar{z}^{n-2}} z + \dots + a_{n,n-1} z^{n-1} \right) \\ &= (1/\bar{z}) P_{n-1}(z, 1/\bar{z}) \end{aligned}$$

where  $P_{n-1}$  is a homogeneous polynomial of degree  $n-1$ .

If  $n \leq -1$  then we get

$$\begin{aligned} c_n(r)e^{in\theta} &= (a_{n,0} + a_{n,1}r^2 + \cdots + a_{n,n-1}r^{2(-n-1)})z^{-|n|} \\ &= \frac{1}{z} \left( a_{n,0} \frac{1}{z^{|n|-1}} + a_{n,1} \frac{1}{z^{|n|-2}} \bar{z} + \cdots + a_{n,n-1} \bar{z}^{n-1} \right) \\ &= (1/z) Q_{|n|-1}(\bar{z}, 1/z) \end{aligned}$$

where  $Q_{|n|-1}$  is a homogeneous polynomial of degree  $|n| - 1$ . Thus, putting  $z = re^{i\theta}$  into the series

$$(1/\bar{z}) \sum_{n=0}^{\infty} P_n(z, 1/\bar{z}) + (1/z) \sum_{n=0}^{\infty} Q_n(\bar{z}, 1/z) \quad (2.1)$$

we get the Fourier series of the function  $e^{i\theta} \mapsto f(re^{i\theta})$ ,  $r_1 \leq r \leq r_2$ .

For each  $t$ ,  $0 < t < 1$ , define the functions  $f_t^+$  and  $f_t^-$  on  $A$  as follows

$$f_t^+(re^{i\theta}) = \sum_{k=1}^{\infty} t^k c_k(r) e^{ik\theta} = (1/\bar{z}) \sum_{j=0}^{\infty} t^{j+1} P_j(z, 1/\bar{z}) \quad (2.2)$$

$$f_t^-(re^{i\theta}) = \sum_{k=-\infty}^{-1} t^{|k|} c_k(r) e^{ik\theta} = (1/z) \sum_{j=0}^{\infty} t^{j+1} Q_j(\bar{z}, 1/z) \quad (2.3)$$

where  $z = re^{i\theta} \in A$  and let

$$f_t(z) = f_t^+(z) + f_t^-(z) \quad (z \in A).$$

Note that for each  $t$ ,  $0 < t < 1$ , both series above converge uniformly on  $A$ . Write

$$\Phi_r(z) = \frac{1}{2\pi i} \int_{b\Delta(r,0)} \frac{f(\zeta) d\zeta}{\zeta - z} \quad (r_1 \leq r \leq r_2)$$

and observe that

$$\left. \begin{aligned} \Phi_r(tre^{i\theta}) &= f_t^+(re^{i\theta}) \quad (r_1 \leq r \leq r_2, 0 < t < 1, \theta \in \mathbb{R}) \\ \Phi_r((1/t)re^{i\theta}) &= -f_t^-(re^{i\theta}) \quad (r_1 \leq r \leq r_2, 0 < t < 1, \theta \in \mathbb{R}) \end{aligned} \right\} \quad (2.4)$$

### 3. Proof of Theorem 1.1

Let  $f$  be a Hölder continuous function on  $A$  which satisfies (1.5) for each  $b\Delta(a, \rho)$  which surrounds the origin. Then there are  $P_n$  and  $Q_n$  as above such that (2.1) with  $z = re^{i\theta}$  is the Fourier series of  $e^{i\theta} \mapsto f(re^{i\theta})$ ,  $r_1 \leq r \leq r_2$ . Using the decomposition (1.1) on each circle  $b\Delta(0, r)$ ,  $r_1 \leq r \leq r_2$ , we can write  $f = f^+ + f^-$  where for each  $r$ ,  $r_1 \leq r \leq r_2$ ,  $f^+|_{b\Delta(0, r)}$  satisfies (1.2) and  $f^-|_{b\Delta(0, r)}$  satisfies (1.3). In fact, the functions  $f^+|_{b\Delta(0, r)}$  and  $-f^-|_{b\Delta(0, r)}$  are the limiting values of  $\Phi_r(z)$  as  $|z| \nearrow r$  or

$|z| \searrow r$ , respectively. Since  $f$  is Hölder continuous on  $A$  it follows that  $f^+$  and  $f^-$  are Hölder continuous on  $A$  [M, Sections 19, 20].

For each  $r$ ,  $r_1 \leq r \leq r_2$ , the function  $\Phi_r$  is the Cauchy integral and hence the Poisson integral of  $f^+|b\Delta(0, r)$  so (2.4) implies that for each  $r$ ,  $r_1 \leq r \leq r_2$ ,  $f_t^+(re^{i\theta})$  converges uniformly in  $\theta$  to  $f^+(re^{i\theta})$  as  $t \nearrow 1$ , and since  $f$  is uniformly continuous on  $A$ , the standard proof of the boundary continuity of the Poisson integral shows that the convergence is uniform also in  $r$ ,  $r_1 \leq r \leq r_2$ . So  $f_t$  converges to  $f^+$  uniformly on  $A$  as  $t \nearrow 1$ . Similarly we show that  $f_t^-$  converges to  $f^-$  uniformly on  $A$  as  $t \nearrow 1$ .

Observe that for each  $b\Delta(a, \rho)$  that surrounds the origin, the restriction of  $1/\bar{z}$  to  $b\Delta(a, \rho)$  satisfies (1.2) and the restriction of  $\bar{z}$  to  $b\Delta(a, \rho)$  has a continuous extension to  $[\mathbb{C} \cup \{\infty\}] \setminus \Delta$  which is holomorphic on  $[\mathbb{C} \cup \{\infty\}] \setminus \bar{\Delta}$ . The uniform convergence of the series (2.2) implies that for each  $t$ ,  $0 < t < 1$ ,  $f_t^+|b\Delta(a, \rho)$  satisfies (1.2) whenever  $b\Delta(a, \rho) \subset A$  surrounds the origin. Similarly, the uniform convergence of the series (2.3) and the multiplication with  $1/z$  imply that for each  $t$ ,  $0 < t < 1$ ,  $f_t^-|b\Delta(a, \rho)$  satisfies (1.3) whenever  $b\Delta(a, \rho) \subset A$  surrounds the origin. The uniform convergence of  $f_t^+$  to  $f^+$  and  $f_t^-$  to  $f^-$  as  $t \nearrow 1$  imply that the analogous statements hold for  $f^+$  and  $f^-$ . This completes the proof.

**Remark 1.** Note that the proof of Theorem 1.1. becomes simpler in the special case when  $f$  is smooth, say of class  $\mathcal{C}^2$  on  $A$ . Recall that putting  $z = re^{i\theta}$  into the series (2.1) we get the Fourier series of the function  $e^{i\theta} \mapsto f(re^{i\theta})$ . Integrating by parts we see that there is a constant  $M < \infty$  such that each of the series in (2.1) is dominated on  $A$  by the series  $\sum_{n=1}^{\infty} Mn^{-2}$  which implies that one can define

$$f^+(z) = (1/\bar{z}) \sum_{n=0}^{\infty} P_n(z, 1/\bar{z}), \quad f^-(z) = (1/z) \sum_{n=0}^{\infty} Q_n(\bar{z}, 1/z) \quad (z \in A)$$

where each of the series converges uniformly on  $A$ .

**Remark 2.** If  $f$  in Theorem 1.1 is Hölder continuous on  $A$  with exponent  $\alpha$ ,  $0 < \alpha < 1$ , then for any  $\beta$ ,  $0 < \beta < \alpha$ , the functions  $f^+$  and  $f^-$  are Hölder continuous on  $A$  with exponent  $\beta$ . This follows from [M, Sections 19, 20].

#### 4. Domains $\Omega^+(A)$ and $\Omega^-(A)$

We list some simple facts about the domain  $\Omega^+(A)$ . Analogous statements hold for  $\Omega^-(A)$ , the image of  $\Omega^+(A)$  under the reflection  $(z, w) \mapsto (\bar{w}, \bar{z})$ . The proofs are elementary, they can be found in [G2].

Recall that  $\Omega^+(A)$  is defined as the union of all  $\Lambda_{a, \rho}^+$  such that  $b\Delta(a, \rho) \subset \text{Int}A$  surrounds the origin.

**Proposition 4.1** *Let  $\gamma = (r_1 + r_2)/2$ . The set  $\Omega^+(A)$  is a disjoint union of all  $\Lambda_{a, \gamma}^+$  such that  $b\Delta(a, \gamma) \subset \text{Int}A$ ; it is an unbounded open connected set whose boundary consists of  $\tilde{A}$  together with the union of all those  $\Lambda_{a, \gamma}^+$  for which  $b\Delta(a, \gamma) \subset A$  is tangent to both  $b\Delta(0, r_1)$  and  $b\Delta(0, r_2)$ . For each  $\zeta \in \text{Int}A$  there are a neighbourhood  $U \subset \Sigma$  of  $(\zeta, \bar{\zeta})$  an open cone  $V$  in  $i\Sigma$ , a real two-plane perpendicular to  $\Sigma$ , and a  $\delta > 0$  such that  $U + (V \cap \delta\mathbb{B}) \subset \Omega^+(A)$ .*

**Proposition 4.2** *If  $\Lambda_{a,\rho}^+ \neq \Lambda_{b,\delta}^+$  then the sets  $\Lambda_{a,\rho}^+$  and  $\Lambda_{b,\delta}^+$  intersect if and only if  $a \neq b$  and one of the circles  $b\Delta(a, \rho)$ ,  $b\Delta(b, \delta)$  surrounds the other. The sets  $\Lambda_{a,\rho}^+$  and  $\Lambda_{b,\delta}^-$  intersect if and only if  $\overline{\Delta}(a, \rho) \cap \overline{\Delta}(b, \delta) = \emptyset$ .*

Proposition 4.2 implies that  $\Omega^+(A) \cap \Omega^-(A) = \emptyset$ .

## 5. Proof of Theorem 1.2

Recall that  $f^+$  is the uniform limit of  $f_t^+$  as  $t \nearrow 1$  where for each  $t$ ,  $0 < t < 1$ ,

$$f_t^+(z) = (1/\bar{z}) \sum_{j=0}^{\infty} t^{j+1} P_j(z, 1/\bar{z}) \quad (z \in A)$$

with the series converging uniformly on  $A$ . It follows that on  $A$ ,  $\bar{z}f^+(z)$  is a uniform limit of a sequence of polynomials in  $z$  and  $1/\bar{z}$ ,

$$\bar{z}f^+(z) = \lim_{m \rightarrow \infty} S_m(z, 1/\bar{z}) \quad (z \in A). \quad (5.1)$$

Now we reason as in [G2]: The functions  $S_m(z, 1/w)$  are bounded and continuous on  $\Lambda_{a,\rho}^+ \cup b\Lambda_{a,\rho}^+$  and holomorphic on  $\Lambda_{a,\rho}^+$  whenever  $b\Delta(a, \rho) \subset A$  surrounds the origin. Since  $\Lambda_{a,\rho}^+$  is biholomorphically equivalent to the punctured disc the maximum principle implies that for each  $(z, w) \in \Lambda_{a,\rho}^+$  we have

$$\begin{aligned} |S_m(z, 1/w) - S_j(z, 1/w)| &\leq \max\{|S_m(\zeta, 1/\eta) - S_j(\zeta, 1/\eta)| : (\zeta, \eta) \in b\Lambda_{a,\rho}^+\} \\ &\leq \max\{|S_m(\zeta, 1/\bar{\zeta}) - S_j(\zeta, 1/\bar{\zeta})| : \zeta \in A\}. \end{aligned}$$

It follows that the sequence  $S_m(z, 1/w)$  converges uniformly on  $\Omega^+(A) \cup b\Omega^+(A)$  to a function  $G^+(z, w)$ . Since each  $S_m(z, 1/w)$  is bounded and continuous on  $\Omega^+(A) \cup b\Omega^+(A)$  and holomorphic on  $\Omega^+(A)$  the same is true for  $G^+$ . Obviously,  $f^+(z) = (1/\bar{z})G^+(z, \bar{z})$  ( $z \in A$ ). In the same way we prove that  $f^-(z) = (1/z)G^-(z, \bar{z})$  ( $z \in A$ ) where  $G^-$  is bounded and continuous on  $\Omega^-(A) \cup b\Omega^-(A)$  and holomorphic on  $\Omega^-(A)$ . This completes the proof.

## 6. Dropping the assumption on Hölder continuity

The map  $\zeta \mapsto \zeta^* = 1/\bar{\zeta}$  is the antiholomorphic reflection across  $b\Delta$  which fixes  $b\Delta$ . Similarly, given  $a \in \mathbb{C}$  and  $\rho > 0$ , the map  $\zeta \mapsto \zeta^* = a + \rho^2/(\bar{\zeta} - \bar{a})$  is the antiholomorphic reflection across  $b\Delta(a, \rho)$  which fixes  $b\Delta(a, \rho)$  and maps a point on a ray emanating from  $a$  at a distance  $\gamma > 0$  from  $a$  to the point on the same ray at a distance  $\rho^2/\gamma$  from  $a$ .

If  $f$  is a continuous function on  $b\Delta(a, \rho)$  which is not necessarily Hölder continuous then the functions  $f^+$  and  $f^-$  defined by (1.5) are well defined away from  $b\Delta(a, \rho)$  but need not have boundary values as we approach  $b\Delta(a, \rho)$ . The following still holds and is well known:

**Lemma 6.1** [Z, Vol. 1, p. 288] *Let  $f$  be a continuous function on  $b\Delta(a, \rho)$ . Define  $f^+$  and  $f^-$  by (1.4). Then the function*

$$z \mapsto \begin{cases} f^+(z) + f^-(z^*) & (z \in \Delta(a, \rho)) \\ f(z) & (z \in b\Delta(a, \rho)) \end{cases}$$

is continuous on  $\overline{\Delta}(a, \rho)$ . In fact, on  $\Delta(a, \rho)$  it coincides with the Poisson integral of  $f$ .

We want to show that a generalization of this holds for continuous functions on the annulus  $A$  with zero means on circles surrounding the origin.

Suppose that  $f$  is a continuous function on  $A$  which satisfies (1.5) whenever  $b\Delta(a, \rho) \subset A$  surrounds the origin. Recall that for each nonnegative integer  $n$  there are homogeneous polynomials  $P_n$  and  $Q_n$  of degree  $n$  such that putting  $z = re^{i\theta}$  into

$$(1/\bar{z}) \sum_{n=0}^{\infty} P_n(z, 1/\bar{z}) + (1/z) \sum_{n=0}^{\infty} Q_n(\bar{z}, 1/z)$$

we get the Fourier series of the function  $e^{i\theta} \mapsto f(re^{i\theta})$ ,  $r_1 \leq r \leq r_2$ . We now show that one can define a holomorphic function  $F^+$  on  $\Omega^+(A)$  by

$$F^+(z, w) = (1/w) \sum_{n=0}^{\infty} P_n(z, 1/w) \quad ((z, w) \in \Omega^+(A)) \quad (6.1)$$

where the series converges uniformly on compacta in  $\Omega^+(A)$  and a holomorphic function  $F^-$  on  $\Omega^-(A)$  by

$$F^-(z, w) = (1/z) \sum_{n=0}^{\infty} Q_n(w, 1/z) \quad ((z, w) \in \Omega^-(A)) \quad (6.2)$$

where the series converges uniformly on compacta in  $\Omega^-(A)$ .

Recall that for each  $t$ ,  $0 < t < 1$ , the series  $(1/\bar{z}) \sum_{j=1}^{\infty} t^{j+1} P_j(z, 1/\bar{z})$  converges uniformly on  $A$  which, by a reasoning similar to the one in Section 5 implies that the function

$$F_t^+(z, w) = (1/w) \sum_{j=0}^{\infty} t^{j+1} P_j(z, 1/w) \quad (6.3)$$

is well defined, bounded and continuous on  $\Omega^+(A) \cup b\Omega^+(A)$  and holomorphic on  $\Omega^+(A)$  since the series (6.3) converges uniformly on  $\Omega^+(A) \cup b\Omega^+(A)$ . We have  $F_t^+(z, \bar{z}) = f_t^+(z)$  ( $z \in A$ ). Similarly, the function

$$F_t^-(z, w) = (1/z) \sum_{j=0}^{\infty} t^{j+1} Q_j(w, 1/z)$$

is well defined, bounded and continuous on  $\Omega^-(A) \cup b\Omega^-(A)$  and holomorphic on  $\Omega^-(A)$  since the series converges uniformly on  $\Omega^-(A) \cup b\Omega^-(A)$ . We have  $F_t^-(z, \bar{z}) = f_t^-(z)$  ( $z \in A$ ).

Using the homogeneity of  $P_j$  we rewrite (6.3) to

$$F_t^+(z, w) = \frac{1}{(w/t)} \sum_{j=0}^{\infty} P_j(tz, 1/(w/t))$$

and we see that (6.1) converges uniformly on  $T_t(\Omega^+(A))$  where  $T_t(z, w) = (tz, w/t)$ . Given a compact set  $K \subset \Omega^+(A)$  there is a  $t$ ,  $0 < t < 1$ , such that  $K \subset T_t(\Omega^+(A))$  so (6.1) converges uniformly on compacta in  $\Omega^+(A)$ . We have

$$F_t^+(z, w) = F(T_t(z, w)) \quad ((z, w) \in T_t^{-1}(\Omega^+(A))).$$

Since  $T_t$  converges uniformly to the identity as  $t \rightarrow 1$  it follows that

$$F^+(z, w) = \lim_{t \rightarrow 1} F_t^+(z, w) \quad ((z, w) \in \Omega^+(A)). \quad (6.4)$$

where the convergence is uniform on compact sets in  $\Omega^+(A)$ . In the same way we see that

$$F^-(z, w) = \lim_{t \rightarrow 1} F_t^-(z, w) \quad ((z, w) \in \Omega^-(A)). \quad (6.5)$$

where the convergence is uniform on compact sets in  $\Omega^-(A)$ .

One can verify that for each  $r$ ,  $r_1 \leq r \leq r_2$ , and for each  $t$ ,  $0 < t < 1$ ,

$$f_t(re^{i\theta}) \equiv \mathcal{P}_r(te^{i\theta}) \quad (\theta \in \mathbb{R})$$

where  $\mathcal{P}_r$  is the Poisson integral of the function  $e^{i\theta} \mapsto f(re^{i\theta})$ . Now,  $\mathcal{P}_r(te^{i\theta}) \rightarrow f(re^{i\theta})$  uniformly in  $\theta$  as  $t \nearrow 1$  and since  $f$  is uniformly continuous on  $A$ , the standard proof of the boundary continuity of the Poisson integral shows that the convergence is uniform also in  $r$ ,  $r_1 \leq r \leq r_2$ . Thus,

$$f_t \rightarrow f \text{ uniformly on } A \text{ as } t \nearrow 1. \quad (6.6)$$

## 7. Continuous functions with zero means on circles

**Theorem 7.1** *Let  $f$  be a continuous function on  $A$  which satisfies (1.5) for each  $b\Delta(a, \rho) \subset A$  which surrounds the origin. There are a holomorphic function  $F^+$  on  $\Omega^+(A)$  and a holomorphic function  $F^-$  on  $\Omega^-(A)$  such that the function*

$$(z, w) \mapsto \begin{cases} F^+(z, w) + F^-(\bar{w}, \bar{z}) & ((z, w) \in \Omega^+(A)) \\ f(z) & (z, \bar{z}) \in \tilde{A} \end{cases} \quad (7.1)$$

*has a bounded continuous extension to  $\Omega^+(A) \cup b\Omega^+(A)$ .*

Thus, for each  $z \in A$  we have

$$f(z) = \lim_{(\xi, \eta) \rightarrow (z, \bar{z}), (\xi, \eta) \in \Omega^+(A)} [F^+(\xi, \eta) + F^-(\bar{\eta}, \bar{\xi})].$$

Theorem 7.1 is the analogue of Lemma 6.1 where  $b\Delta(a, \rho)$  is replaced by  $\tilde{A}$  and  $\Delta(a, \rho)$  is replaced by  $\Omega^+(A)$ . The function (7.1) is the analogue of the Poisson integral of  $f$ .



It is the bounded continuous extension of  $F(z, \bar{z}) = f(z)$  to  $\Omega^+(A) \cup b\Omega^+(A)$  which is pluriharmonic on  $\Omega^+(A)$ .

**Proof of Theorem 7.1.** For each  $t$ ,  $0 < t < 1$ , define

$$\Psi_t(z, w) = F_t^+(z, w) + F_t^-(\bar{w}, \bar{z}) \quad ((z, w) \in \Omega^+(A) \cup b\Omega^+(A)),$$

where  $F_t^+$  and  $F_t^-$  are as in Section 6. The properties of  $F_t^+$  and  $F_t^-$  imply that  $\Psi_t$  is bounded and continuous on  $\Omega^+(A) \cup b\Omega^+(A)$ , pluriharmonic on  $\Omega^+(A)$  and satisfies  $\Psi_t(z, \bar{z}) = f_t^+(z) + f_t^-(z) = f_t(z)$  ( $z \in A$ ). It follows that for each  $t, s$ ,  $0 < t < 1$ ,  $0 < s < 1$  the function

$$(z, w) \mapsto |\Psi_t(z, w) - \Psi_s(z, w)|, \tag{7.2}$$

restricted to  $\Lambda_{a,\rho} \cup b\Lambda_{a,\rho}$ , attains its maximum on  $b\Lambda_{a,\rho}$  whenever  $b\Delta(a, \rho) \subset A$  surrounds the origin. This is so since  $\Lambda_{a,\rho}$  is biholomorphically equivalent to the punctured disc and since isolated singularities are removable for bounded harmonic functions. Thus, the function (7.2) attains its maximum on  $\Omega^+(A) \cup b\Omega^+(A)$  on  $\tilde{A}$ . By (6.6) the restrictions of functions  $\Psi_t$  to  $\tilde{A}$  converge uniformly as  $t \nearrow 1$  which, by the preceding discussion implies that as  $t \nearrow 1$  the functions  $\Psi_t$  converge uniformly on  $\Omega^+(A) \cup b\Omega^+(A)$  to a bounded continuous function  $\Psi$  which is pluriharmonic on  $\Omega^+(A)$  and which satisfies  $\Psi(z, \bar{z}) = f(z)$  ( $z \in A$ ). Now (6.4) and (6.5) imply that  $\Psi(z, w) = F^+(z, w) + F^-(z, w)$  ( $(z, w) \in \Omega^+(A)$ ) where  $F^+$  and  $F^-$  are given by (6.1) and (6.2). This completes the proof.

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